LEARNING THE DYNAMIC SIR MODEL

AN OPTIMAL CONTROL APPROACH

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SIR Models

THE MODEL i

A classical deterministic approach to model an epidemic is the *susceptible, infectious, removed* (SIR) model.

A population of n individuals is divided into

- $x(t) = (\#$ of susceptible at time $t)$;
- $y(t) = (\#$ of infectious at time $t)$;
- $z(t) = (\#$ of removed at time $t)$.

The hypothesis of constant population gives

$$x(t) + y(t) + z(t) = n.$$

THE MODEL ii

The dynamics of the epidemics then can be described by the following system of ODEs[1]:

$$\begin{cases} x'(t) = -\beta x(t)y(t); \\ y'(t) = \beta x(t)y(t) - \gamma y(t); \\ z'(t) = \gamma y(t). \end{cases}$$

- $\beta > 0 \to$ infection rate;
- $\gamma > 0 \to$ removal rate.
- $R_0 := \beta/\gamma \to$ reproduction number.

[1] Norman TJ Bailey et al. *The mathematical theory of infectious diseases and its applications*. Charles Griffin & Company Ltd, 5a Crendon Street, High Wycombe, Bucks HP13 6LE., 1975.

DYNAMIC SIR

A more general epidemiological model is the *dynamic SIR*, where we allow to have time-varying infection rate and removal rate:

$$\beta \to \beta(t), \qquad \gamma \to \gamma(t).$$

We also add a new state:

$$w(t) = \delta(t) z(t),$$

where $w(t)$ is the number of deceased people at time t and $\delta(t)$ is the fatality rate.

The problem

GENERIC STATEMENT

Problem

Learning $\beta(t)$, $\gamma(t)$ and $\delta(t)$ with supervisions on the number of deceased assuming that

$$w(t) = \delta(t)z(t),$$

where $w(t)$ is the number of deceased people at time t.

A MORE PRECISE FORMULATION

Let $u(t) = (\beta(t), \gamma(t), \delta(t))$ and

- $w(t; u) := \delta(t) z(t; u)$ where $z(t; u)$ is the solution of the dynamic SIR model when we choose u as time-dependent rates;
- Let $\hat{w}(t)$ be the supervision at time t.

Then we search for the estimate u^* such that

$$u^* = \arg\min_u F(u),$$

where

$$F(u) := \frac{1}{T} \int_0^T \frac{1}{2} \big(w(t; u) - \hat{w}(t) \big)^2 \, dt.$$

In order to find the minimum of the functional $u \mapsto F(u)$ we use gradient descent[2]; starting from a initial point $u_0(t)$ we update our estimate following

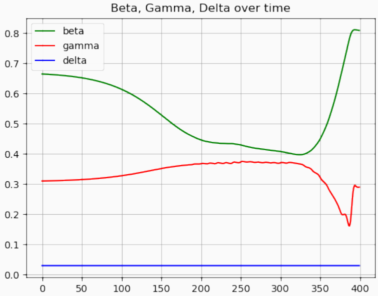
$$u_{k+1}(t) = u_k(t) - \tau \partial F(u_k(t)).$$

[2] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. *Gradient flows: in metric spaces and in the space of probability measures*. Springer Science & Business Media, 2008.

VANISHING GRADIENTS

When using the above update rule, we incur in something similar to the vanishing gradients. This is due to the fact that $\hat{w}$ is computed through numeric integration.

Therefore, values of $u(t)$ with higher t have less impact on the final solution of the integration, and thus have smaller gradients and are hardly moved from their initial values (at epoch $= 0$).



MOMENTUM TERM

To contrast the vanishing gradients we added a ”*momentum*” term in the update rule.

$$u_t^{k+1} = \begin{cases} u_t^k - \alpha_t \nabla_t f(u^k) & \text{if} \quad t = 0 \\ u_t^k - \alpha_t \nabla_t f(u^k) + \mu_t (u_{t-1}^{k+1} - u_{t-1}^k) & \text{if} \quad t > 0 \end{cases} .$$

$$\mu(t) = sigmoid(mt)$$

This term manages to distribute the higher gradients of early u throughout the whole function.

REGULARIZATION

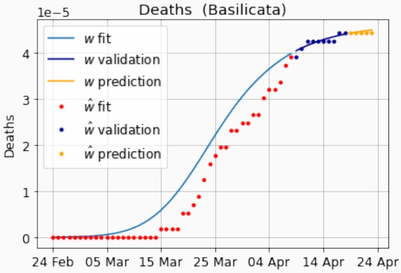
In order to have a well posed problem we added to the functional F a regularization so that what we are actually looking for solution to the problem

$$u^* = \arg\min_u F(u) + R(u),$$

where $R(u)$ is a functional that enforces smoothness on u; a natural choice is

$$R(u) = \int_0^T \frac{|u'|^2}{2}.$$

Experiments


Deaths (Basilicata)
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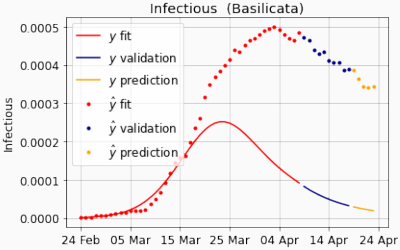
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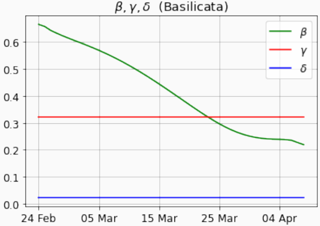
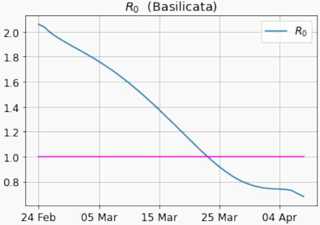
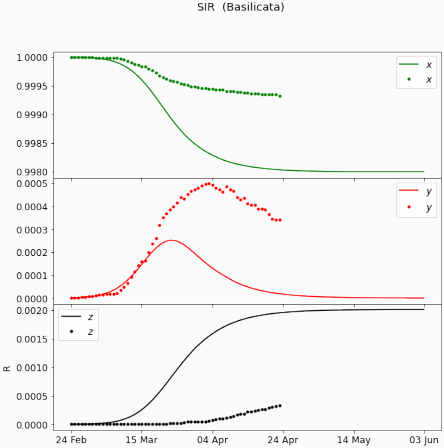
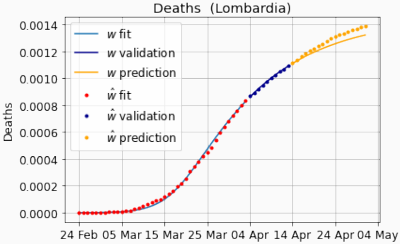
SIR DYNAMICS - BASILICATA

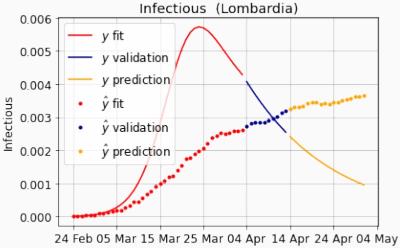






FIT - LOMBARDIA





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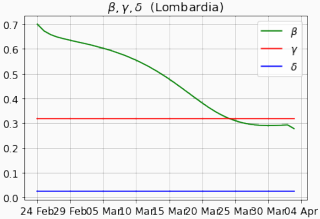

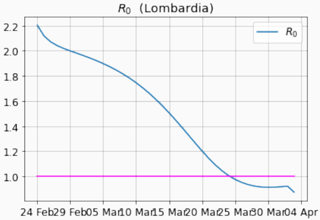

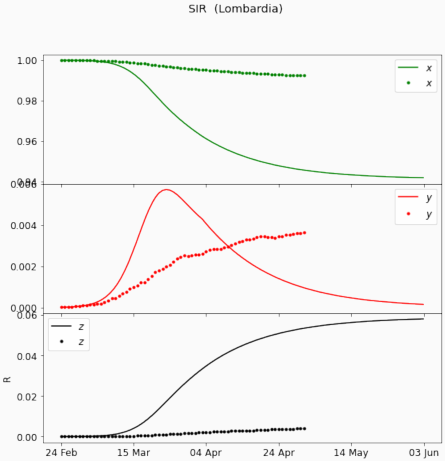

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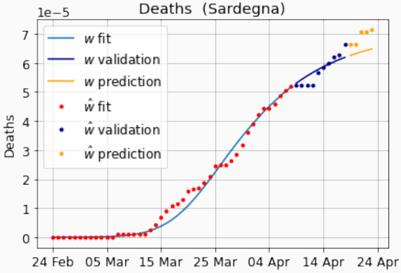

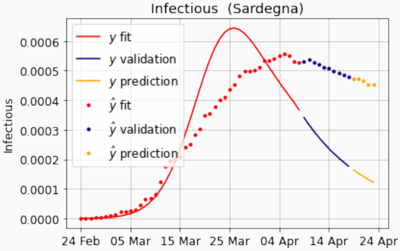

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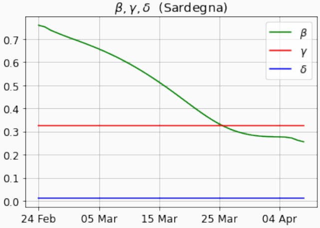
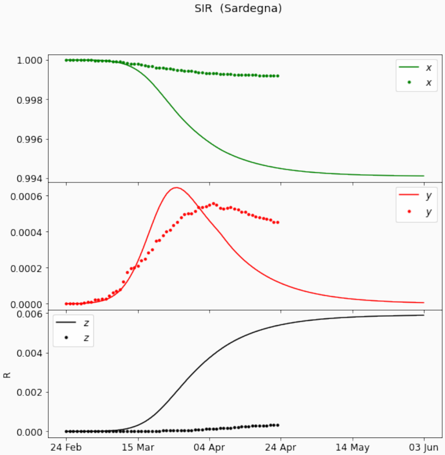
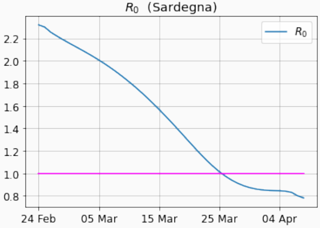
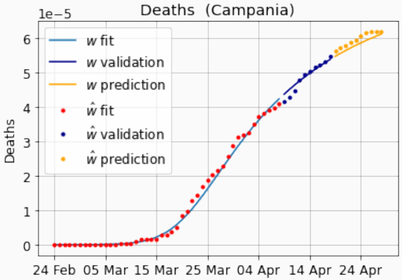
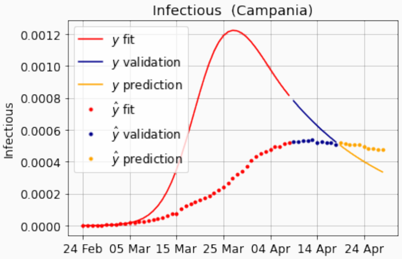




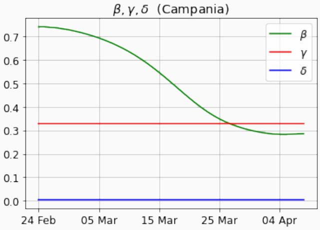


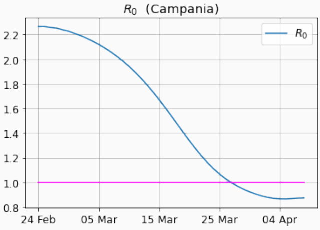
FIT - CAMPANIA

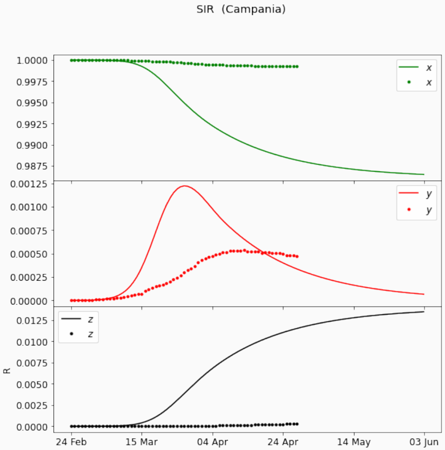




SIR DYNAMICS - CAMPANIA







Thank you for listening!

References

Ambrosio, Luigi, Nicola Gigli, and Giuseppe Savaré. *Gradient flows: in metric spaces and in the space of probability measures*. Springer Science & Business Media, 2008.

Bailey, Norman TJ et al. *The mathematical theory of infectious diseases and its applications*. Charles Griffin & Company Ltd, 5a Crendon Street, High Wycombe, Bucks HP13 6LE., 1975.

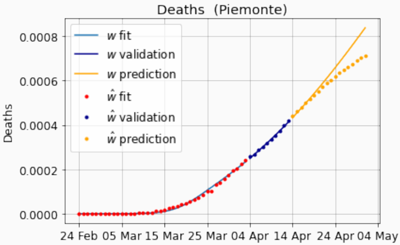
BACKUP SLIDES–NORMALIZATION

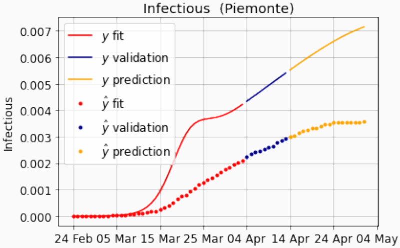
An obvious observation is that if we rescale all the variables by the total population as follows

$$\begin{aligned} &x \to x/n, \quad y \to y/n, \quad z \to z/n; \\ &\beta \to n\beta, \quad \gamma \to \gamma, \end{aligned}$$

we can work with a renormalized SIR where $x + y + z = 1$.

BACKUP SLIDES – FIT - PIEMONTE





BACKUP SLIDES – SIR DYNAMICS - PIEMONTE